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exponents respectively,

$$(-1)^{n-\sigma} (n_1-1)! (n_2-1)! \dots (n_\sigma-1)! s_{\lambda_{n_1}} s_{\lambda_{n_2}} \dots s_{\lambda_{n_\sigma}},$$

a result that agrees with that obtained in a somewhat different way on page 8 of the German translation of Faa di Bruno's *Formes Binaires*.

*Erlangen, Bavaria, 4 May, 1898.*

## CONVEX SURFACE AND VOLUME OF CONICAL UNGULÆ.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let  $BD=c$ ,  $DE=a$ ,  $\tan DBC=n$ ,  $\cot FEC=m$ . Also let  $DH=h$ ,  $DC=R$ ,  $HF=r$ ,  $EC=d$ , then  $c=\frac{Rh}{R-r}$ ,  $n=\frac{R-r}{h}$ ,  $a=R-d$ ,  $m=\frac{r-R+d}{h}$ .

Then  $x^2+z^2=n^2(c-y)^2$ , is the equation of the cone, and  $x=my+a$ , is the equation of the plane.

$$\left(\frac{dz}{dx}\right)^2 = \frac{x^2}{z^2}, \quad \left(\frac{dz}{dy}\right)^2 = \frac{n^4(c-y)^2}{z^2};$$

$$\begin{aligned} \therefore \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} \\ = \frac{n(c-y)\sqrt{1+n^2}}{\sqrt{[n^2(c-y)^2 - x^2]}}. \end{aligned}$$

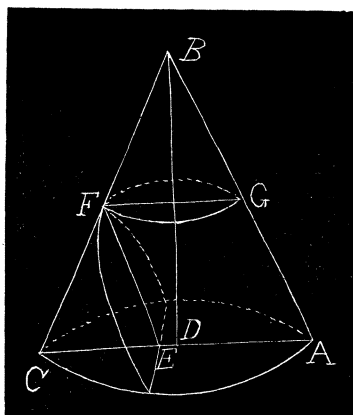
The limits of  $x$  are  $my+a=x_2$  and  $n(c-y)=x_1$ ; of  $y$ , 0 and  $\frac{nc-a}{m+n}=y'$ .

$$\therefore S = 2 \int_0^{y'} \int_{x_2}^{x_1} \frac{n(c-y)\sqrt{1+n^2}}{\sqrt{[n^2(c-y)^2 - x^2]}}$$

$$= n\sqrt{1+n^2} \int_0^{y'} \left[ \pi - 2\sin^{-1} \left( \frac{a+my}{n(c-y)} \right) \right] (c-y) dy$$

$$= nc^2\sqrt{1+n^2} \left[ \frac{1}{2}\pi - \sin^{-1} \left( \frac{a}{nc} \right) \right]$$

$$- n(a+mc)\sqrt{1+n^2} \int_a^{y'} \frac{(c-y)dy}{\sqrt{[n^2(c-y)^2 - (a+my)^2]}}.$$



$$n^2(c-y)^2 - (a+my)^2 = [(mnc+na)^2 - (n^2c+am+m^2y-n^2y)^2]/(m^2-n^2).$$

When  $m^2 > n^2$

$$= (nc+a)(nc-a-2ny).$$

When  $m^2 = n^2$

$$= [(n^2c+am+m^2y-n^2y)^2 - (mnc+na)^2]/(n^2-m^2)$$

When  $m^2 < n^2$

$$\text{Let } u = n^2c + am + m^2y - n^2y.$$

Then the limits of  $u$  are  $n^2c + am = u_2$  and  $mnc + an = u_1$ . When  $m^2 > n^2$

$$\begin{aligned} n(a+mc)\sqrt[4]{1+n^2} \int_0^{y'} \frac{(c-y)dy}{\sqrt[4]{[n^2(c-y)^2 - (a+my)^2]}} \\ = \frac{n(a+mc)\sqrt[4]{1+n^2}}{\sqrt[4]{(m^2-n^2)^3}} \int_{u_2}^{u_1} \frac{(cm^2+am-u)du}{\sqrt[4]{(an+mcn)^2-u^2}} \\ = \frac{\pi nm(a+mc)^2\sqrt[4]{1+n^2}}{2\sqrt[4]{(m^2-n^2)^3}} - \frac{nm(a+mc)^2\sqrt[4]{1+n^2}}{\sqrt[4]{(m^2-n^2)^3}} \sin^{-1}\left(\frac{n^2c+am}{an+mc}\right) \\ - \frac{n(a+mc)\sqrt[4]{1+n^2}}{\sqrt[4]{(m^2-n^2)^3}} \frac{1}{\sqrt[4]{(an+mcn)^2 - (n^2c+am)^2}}. \end{aligned}$$

$$\begin{aligned} \therefore S = n\sqrt[4]{1+n^2} \left[ \frac{\pi c^2}{2} - \frac{\pi m(a+mc)^2}{2\sqrt[4]{(m^2-n^2)^3}} - c^2 \sin^{-1}\left(\frac{a}{nc}\right) \right. \\ \left. + \frac{m(a+mc)^2}{\sqrt[4]{(m^2-n^2)^3}} \sin^{-1}\left(\frac{n^2c+am}{an+mc}\right) \right. \\ \left. + \frac{(a+mc)}{\sqrt[4]{(m^2-n^2)^3}} \frac{1}{\sqrt[4]{(an+mcn)^2 - (n^2c+am)^2}} \right]. \end{aligned}$$

$$\begin{aligned} S = \frac{\sqrt[4]{h^2 + (R-r)^2}}{R-r} \left[ \frac{\pi R^2}{2} - \frac{\pi r^2 d(r-R+d)}{2\sqrt[4]{d(d+2r-2R)^3}} \right. \\ \left. + \frac{r^2 d(r-R+d)}{\sqrt[4]{d(d+2r-2R)^3}} \sin^{-1}\left(\frac{2R-r-d}{r}\right) \right. \\ \left. - R^2 \sin^{-1}\left(\frac{R-d}{R}\right) + \frac{rd(R-r)}{\sqrt[4]{d(d+2r-2R)^3}} \sqrt[4]{r^2 - (2R-r-d)^2} \right]. \end{aligned}$$

$$\text{Let } d=2R. \therefore S = \frac{\pi\sqrt[4]{h^2 + (R-r)^2}}{R-r} \left( R^2 - \frac{1}{2}\sqrt[4]{Rr(R+r)} \right).$$

When  $m^2 = n^2$ ,  $y' = (nc - a)/2n$ .

$$n(a + mc)\sqrt{(1 + n^2)} \int_0^{y'} \frac{(c - y)dy}{\sqrt{[n^2(c - y)^2 - (a + my)^2]}}$$

$$= n\sqrt{a + nc} \sqrt{1 + n^2} \int_0^{y'} \frac{(c - y)dy}{\sqrt{[nc - a - 2ny]}} = \frac{(2nc + a)\sqrt{[(n^2c^2 - a^2)(1 + n^2)]}}{3n}.$$

$$\therefore S = \sqrt{(1 + n^2)} \left[ \frac{\pi c^2 n}{2} - nc^2 \sin^{-1} \left( \frac{a}{nc} \right) - \frac{(2nc + a)\sqrt{(n^2c^2 - a^2)}}{3n} \right].$$

$$\therefore S = \frac{\sqrt{[h^2 + (R - r)^2]}}{R - r} \left[ \frac{\pi R^2}{2} - R^2 \sin^{-1} \left( \frac{R - d}{R} \right) - \frac{1}{8}(3R - d)\sqrt{d(2R - d)} \right].$$

But  $d = 2(R - r)$ .

$$\therefore S = \frac{\sqrt{[h^2 + (R - r)^2]}}{R - r} \left[ \frac{\pi R^2}{2} - R^2 \sin^{-1} \left( \frac{2r - R}{R} \right) - \frac{3}{8}(R + 2r)\sqrt{r(R - r)} \right].$$

When  $m^2 < n^2$ ,

$$n(a + mc)\sqrt{(1 + n^2)} \int_0^{y'} \frac{(c - y)dy}{\sqrt{[n^2(c - y)^2 - (a + my)^2]}}$$

$$= \frac{n(a + mc)\sqrt{(1 + n^2)}}{\sqrt{[(n^2 - m^2)^3]}} \int_{u_2}^{u_1} \frac{(cn^2 + am - u)du}{\sqrt{[u^2 - (mnc + an)^2]}}$$

$$= \frac{n(a + mc)\sqrt{(1 + n^2)}}{\sqrt{[(n^2 - m^2)^3]}} \sqrt{(n^2c + am)^2 - (mnc + an)^2}$$

$$- \frac{nm(a + mc)^2\sqrt{(1 + n^2)}}{\sqrt{[(n^2 - m^2)^3]}} \log \left( \frac{n^2c + am + \sqrt{[(n^2c + am)^2 - (mnc + an)^2]}}{mnc + an} \right).$$

$$\therefore S = n\sqrt{(1 + n^2)} \left[ \frac{\pi c^2}{2} - c^2 \sin^{-1} \left( \frac{a}{nc} \right) \right.$$

$$\left. - \frac{(a + mc)}{\sqrt{[(n^2 - m^2)^3]}} \sqrt{(n^2c + am)^2 - (mnc + an)^2} \right.$$

$$\left. + \frac{m(a + mc)^2}{\sqrt{[(n^2 - m^2)^3]}} \log \left( \frac{n^2c + am + \sqrt{[(n^2c + am)^2 - (mnc + an)^2]}}{mnc + an} \right) \right].$$

$$\therefore S = \frac{\sqrt{[h^2 + (R-r)^2]}}{R-r} \left[ \frac{\pi R^2}{2} - R^2 \sin^{-1} \left( \frac{R-d}{R} \right) \right. \\ \left. - \frac{rd(R-r)}{\sqrt{[d(2R-d-2r)^2]}} \sqrt{[(2R-r-d)^2 - r^2]} \right. \\ \left. + \frac{r^2 d(r-R+d)}{\sqrt{[d(2R-d-2r)^2]}} \log \left( \frac{2R-r-d + \sqrt{[(2R-r-d)^2 - r^2]}}{r} \right) \right].$$

For volume,

$$V = 2 \int_0^{y'} \int_{x_2}^{x_1} \sqrt{[n^2(c-y)^2 - x^2]} dy dx \\ = \int_0^{y'} \left[ \frac{1}{2} \pi n^2(c-y)^2 - n^2(c-y)^2 \sin^{-1} \left( \frac{a+my}{n(c-y)} \right) \right. \\ \left. - (a+my) \sqrt{[n^2(c-y)^2 - (a+my)^2]} \right] dy = \frac{1}{2} \pi n^2 c^3 - \frac{1}{3} n^2 c^3 \sin^{-1} \left( \frac{a}{nc} \right) \\ - \frac{1}{2} n^2(a+mc) \int_0^{y'} \frac{(c-y)^2 dy}{\sqrt{[n^2(c-y)^2 - (a+my)^2]}} \\ - \int_0^{y'} (a+my) \sqrt{[n^2(c-y)^2 - (a+my)^2]} dy.$$

When  $m^2 > n^2$ ,

$$\frac{1}{2} n^2(a+mc) \int_0^{y'} \frac{(c-y)^2 dy}{\sqrt{[n^2(c-y)^2 - (a+my)^2]}} \\ = \frac{n^2(a+mc)}{3\sqrt{[(m^2 - n^2)^3]}} \int_{u_2}^{u_1} \frac{(cm^2 + am - u)^2 du}{\sqrt{[(an + mcn)^2 - u^2]}} \\ = \frac{n^2(a+mc)}{6\sqrt{[(m^2 - n^2)^3]}} \left[ \frac{\pi(a+mc)^2(2m^2 + n^2)}{2} \right. \\ \left. - (a+mc)^2(2m^2 + n^2) \sin^{-1} \left( \frac{n^2 c + am}{an + nmc} \right) \right. \\ \left. + (n^2 c - 4cm^2 - 3am) \sqrt{[(an + nmc)^2 - (am + n^2 c)^2]} \right].$$

$$\int_0^{y'} (a+my) \sqrt{[n^2(c-y)^2 - (a+my)^2]} dy \\ = \frac{1}{\sqrt{[(m^2 - n^2)^3]}} \int_{u_2}^{u_1} (mn - an^2 - n^2 cm) \sqrt{[(an + ncm)^2 - u^2]} du$$

$$= \frac{1}{\sqrt{(m^2 - n^2)^3}} \left[ \frac{1}{3} m [(an + cmn)^2 - (n^2 c + am)]^{\frac{3}{2}} \right. \\ \left. + \frac{n^4 (a + cm)^3}{2} \sin^{-1} \left( \frac{n^2 c + am}{an + cmn} \right) - \frac{\pi n^4 (a + cm)^2}{4} \right. \\ \left. + \frac{n^2 (a + mc)(am + n^2 c)}{2} \sqrt{[(an + cmn)^2 - (am + n^2 c)^2]} \right].$$

$$\therefore V = \frac{1}{6} \pi n^2 c^3 - \frac{1}{3} n^2 c^3 \sin^{-1}(a/n c) - \frac{\pi n^2 (a + cm)^3}{6 \sqrt{(m^2 - n^2)^3}} \\ + \frac{n^2 (a + cm)^3}{3 \sqrt{(m^2 - n^2)^3}} \sin^{-1} \left( \frac{am + n^2 c}{an + cmn} \right) \\ + \frac{2acn^2 + a^2 m + n^2 c^2 m}{3 \sqrt{(m^2 - n^2)^3}} \sqrt{[(an + cmn)^2 - (am + n^2 c)^2]}.$$

$$\therefore V = \frac{R^3 h}{3(R-r)} \left[ \frac{1}{2} \pi - \sin^{-1} \left( \frac{R-d}{R} \right) \right] \\ - \frac{hr^3 d}{3(R-r) \sqrt{[d(d+2r-2R)^3]}} \left[ \frac{1}{2} \pi - \sin^{-1} \left( \frac{2R-r-d}{r} \right) \right] \\ + \frac{h[2Rd(R-d) + d^2(r-R+d)]}{3 \sqrt{[d(d+2r-2R)^3]}} \sqrt{[r^2 - (2R-r-d)^2]}.$$

$$\text{Let } d=2R. \quad \therefore V = \frac{\pi R h}{3(R-r)} [R^2 - r \sqrt{(Rr)}].$$

$$\text{When } m^2 = n^2, \quad y' = [(nc - a)/2n].$$

$$\frac{1}{3} n^2 (a + mc) \int_0^{y'} \frac{(c-y) dy}{\sqrt{[n^2 (c-y)^2 - (a + my)^2]}} \\ = \frac{1}{3} n^2 \sqrt{[a + nc]} \int_0^{y'} \frac{(c-y)^2 dy}{\sqrt{[nc - a - 2ny]}} = \frac{\sqrt{(n^2 c^2 - a^2)} (7n^2 c^2 + 6anc + 2a^2)}{45n}.$$

$$\int_0^{y'} (a + my) \sqrt{[n^2 (c-y)^2 - (a + my)^2]} dy \\ = \sqrt{[nc + a]} \int_0^{y'} (a + ny) \sqrt{[nc - a^2 - 2ny]} dy = \frac{(4a + nc)(nc - a) \sqrt{(n^2 c^2 - a^2)}}{15n}.$$

$$\therefore V = \frac{1}{6} \pi n^2 c^3 - \frac{1}{3} n^2 c^3 \sin^{-1}(a/n c) - \frac{(2n^2 c^2 + 3anc - 2a^2) \sqrt{(n^2 c^2 - a^2)}}{9n}.$$

$$\therefore V = \frac{R^3 h}{3(R-r)} \left[ \frac{1}{2} \pi - \sin^{-1} \left( \frac{R-d}{R} \right) \right] - \frac{h(3R^2 + Rd - 2d^2)}{9(R-r)} \sqrt{[d(2R-d)]}.$$

But  $d=2R-2r$ .

$$\therefore V = \frac{h}{3(R-r)} \left[ \frac{\pi R^3}{2} - R^3 \sin^{-1} \left( \frac{2r-R}{R} \right) + \frac{2(3R^2-14Rr+8r^2)}{3} \sqrt{r(R-r)} \right].$$

When  $m^2 < n^2$ ,

$$\begin{aligned} \frac{1}{3} n^2 (a+mc) \int_0^{y'} \frac{(c-y)^2 dy}{\sqrt{[n^2(c-y)^2 - (a+my)^2]}} \\ = -\frac{n^2(a+mc)}{3\sqrt{(n^2-m^2)^3}} \int_{u_1}^{u_2} \frac{(cm^2+am-u)^2 du}{\sqrt{[u^2 - (an+mnc)^2]}} \\ = -\frac{n^2(a+mc)}{6\sqrt{(n^2-m^2)^3}} \left[ (4cm^2+3am-n^2c) \sqrt{(n^2c+am)^2 - (an+mnc)^2} \right. \\ \left. - (2m^2+n^2)(a+mc)^2 \log \left( \frac{n^2c+am + \sqrt{(n^2c+am)^2 - (an+mnc)^2}}{an+mnc} \right) \right]. \end{aligned}$$

$$\begin{aligned} \int_0^{y'} (a+my) \sqrt{[n^2(c-y)^2 - (a+my)^2]} dy \\ = \frac{1}{\sqrt{(n^2-m^2)^3}} \int_{u_1}^{u_2} (mu - an^2 - n^2cm) \sqrt{[u^2 - (an+mnc)^2]} du \\ = \frac{1}{\sqrt{(n^2-m^2)^3}} \left[ \frac{n^2(a+mc)(am+n^2c)}{2} \sqrt{[(n^2c+am)^2 - (an+mnc)^2]} \right. \\ \left. - \frac{1}{3} m [(n^2c+am)^2 - (an+mnc)^2]^{\frac{3}{2}} \right. \\ \left. - \frac{n^4(a+cm)^3}{2} \log \left( \frac{n^2c+am + \sqrt{(n^2c+am)^2 - (an+mnc)^2}}{an+mnc} \right) \right]. \end{aligned}$$

$$\begin{aligned} \therefore V = \frac{1}{3} \pi n^2 c^3 - \frac{1}{3} n^2 c^3 \sin^{-1} \left( \frac{a}{nc} \right) \\ + \frac{n^2(a+mc)^3}{3\sqrt{(n^2-m^2)^3}} \log \left( \frac{n^2c+am + \sqrt{(n^2c+am)^2 - (an+mnc)^2}}{an+mnc} \right) \\ - \frac{2n^2ac+am^2+n^2c^2m}{3\sqrt{(n^2-m^2)^3}} \sqrt{[(n^2c+am)^2 - (an+mnc)^2]}. \end{aligned}$$

$$\begin{aligned} \therefore V = \frac{R^3 h}{3(R+r)} \left[ \frac{1}{2} \pi - \sin^{-1} \left( \frac{R-d}{R} \right) \right] \\ + \frac{hr^3 d}{3(R-r) \sqrt{[d(2R-d-2r)^3]}} \log \left( \frac{2R-r-d + \sqrt{(2R-r-d)^2 - r^2}}{r} \right) \\ - \frac{h[2Rd(R-d) + d^2(r-R+d)]}{3\sqrt{[d(2R-d-2r)^3]}} \sqrt{[(2R-r-d)^2 - r^2]}. \end{aligned}$$

Let  $m=0$ .

$$\begin{aligned}\therefore V &= \frac{1}{3}n^2c^3 \left[ \frac{1}{2}\pi - \sin^{-1} \left( \frac{a}{nc} \right) \right] \\ &+ \frac{a^3}{3n} \log \left( \frac{nc + \sqrt{[n^2c^2 - a^2]}}{a} \right) - \frac{2}{3}ac\sqrt{[n^2c^2 - a^2]} \\ &= \frac{1}{3}c \left[ \frac{\pi R^2}{2} - \sin^{-1} \left( \frac{a}{R} \right) + a^3 \log \left( \frac{R + \sqrt{[R^2 - a^2]}}{a} \right) - 2a\sqrt{[R^2 - a^2]} \right] \\ &= \frac{Rh}{3(R-r)} \left[ \frac{\pi R^2}{2} - R^2 \sin^{-1}(r/R) + \frac{Rr^3}{R-r} \log \left( \frac{R + \sqrt{[R^2 - r^2]}}{r} \right) - 2r\sqrt{[R^2 - r^2]} \right]\end{aligned}$$

## SOLUTIONS OF PROBLEMS.

### ARITHMETIC.

93. Proposed by **RAYMOND D. SMITH**, Tiffin, Ohio.

A barn 20 feet square is standing in a pasture, and a horse is tied to one corner of it with a rope 50 feet long. Over how much land can he graze?

I. Solution by **B. F. FINKEL**, M. Sc., M. A., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let  $ABCD$  be the barn, side  $AB=AD=20$  feet;  $A$  the corner to which the horse is tied; and  $AF=AG=50$  feet, the length of the rope.

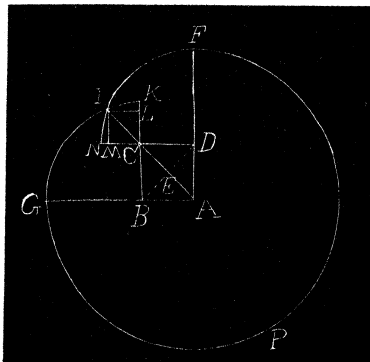
Then  $DI=BI=30$  feet;  $AC=DB=20\sqrt{2}$  feet;  $EI=\sqrt{[DI^2-DE^2]}=\sqrt{[30^2-(10\sqrt{2})^2]}$  feet  $=10\sqrt{7}$  feet;  $CI=EI-EC=10\sqrt{7}$  feet  $-10\sqrt{2}$  feet  $=10(\sqrt{7}-\sqrt{2})$  feet;  $CL=CM=\sqrt{[CI^2/2]}=\frac{1}{2}CI\sqrt{2}=5(\sqrt{14}-2)$  feet;  $KL=CK-CL=10$  feet  $-5(\sqrt{14}-2)$  feet  $=5(4-\sqrt{14})$  feet; and chord  $KI=\text{chord } IN=\sqrt{[KL^2+IL^2]}$ .

$\sqrt{[25(4-\sqrt{14})^2+25^2(\sqrt{14}-2)^2]}$  feet  $=10\sqrt{[3(4-\sqrt{14})]}$  feet.

$2 \text{ arc } IK = \frac{1}{3}(8 \text{ chord } KI - 2IL^*)$   
 $= \frac{1}{3}\{80\sqrt{[3(4-\sqrt{14})]} - 20(\sqrt{7}-\sqrt{2})\}$  feet  $= \frac{2}{3}\{4\sqrt{[3(4-\sqrt{14})]} - (\sqrt{7}-\sqrt{2})\}$  feet.

The area over which the horse can graze  $= FAGPF + \text{sector } FDI + \text{sector } IBG + \text{triangle } DCI + \text{triangle } BCI = FAGPF + 2 \text{ sector } FDI + 2 \text{ triangle } DCI = FAGPF + 2(\text{quadrant } FDN - \text{sector } IDN) + 2 \text{ triangle } DCI$ .

But area of  $FAGPF = \frac{3}{4}\pi AF^2 = 1875\pi$ ;



\*See *Williamson's Differential Calculus*, pages 84-85, for a proof of this rule. The discovery of this important approximation is due to Huygens. The length of an arc of  $30^\circ$  on a circle of radius 100,000 differs from the true value, assuming  $\pi=3.141592$ , by about 2 inches. The formula is  $\text{arc}=\frac{1}{3}(8B-A)$  when  $B$  is the chord of half the arc and  $A$  is chord of the arc.